Convective Diffusion in Rotating Disk Systems with an Imperfect Semipermeable Interface

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Solutions to the momentum and diffusion equations are obtained for rotating disk systems with an imperfect semipermeable interface, with direct application made to the reverse osmosis or hyperfiltration process of salt water purification. The equations are solved exactly, and a new technique for solving the momentum equations is described. An approximate solution to the diffusion equation is also obtained which is also applicable to the energy equation, and is shown to be accurate for Prandtl and Schmidt numbers $\geqq 1$, for a wide range of interfacial mass transfer, for all wedge-type flows as well as the rotating disk system.

One objective of this paper is to discuss the process of reverse osmosis, or hyperfiltration, in a rotating disk system; both exact and very accurate approximate solutions will be presented. The approximate solution developed is similar to well known solutions applicable for high Schmidt and/or Prandtl number systems. The solution developed herein, however, is applicable for lower Schmidt and Prandtl numbers (~ 1) and for a wider range of system geometries, including interfacial mass transfer, as will be demonstrated by application to flat plate, rotating disk and plane stagnation flow systems.

A great deal of time, effort, and money is currently being expended to find means of insuring an adequate supply of pure water for future consumption. Among the potentially important processes for saline water purification, and especially appealing because of its simplicity, is that of reverse osmosis or hyperfiltration. This process consists essentially of separating pure water from a saline solution by a membrane which is permeable to water but not salt, and by maintaining the pressure in the saline solution sufficiently greater than that in the pure water so as to overcome the osmotic pressure, and to cause water to flow through the membrane from the saline side to the pure water side. A major problem in this process is that the rejected salt accumulates near the membrane surface, thereby increasing the osmotic pressure and decreasing the driving force for pure water production. Analytical studies have been published which predict the polarization and water production to be expected in a constant pressure cell (*11*), in turbulent flow in round tube systems (*13*) and between concentric rotating cylinders (*14*), in laminar flow in parallel plate systems (*2, 4, 5, 13*), in tubes and annuli (*6, 13*) and in plane stagnation flow (*18*).

A significant advantage of the last system mentioned above is that the defining equations may be expressed in terms of a single independent variable, which greatly simplifies the analysis. Further, incomplete salt rejection may be easily and exactly treated since the salt concentration at the membrane surface is a constant, which is not true of the other systems mentioned above. However, in real systems, such as a cylinder in cross flow, the area described by the stagnation flow equations may not be large. Another system having the same advantages is that of a rotating disk in an infinite fluid. Additionally, this case represents a fully three-dimensional flow that affords an exact solution of the Navier-Stokes equation (*12*), and in this case the equations are applicable over most of the surface of a finite disk. A further advantage of the rotating disk is that motion is induced by the disk itself and not by bulk movement of the fluid, thus eliminating considerable pumping and plumbing.

As a consequence of both analytical and experimental advantages cited above, rotating disk systems have received considerable attention in studies of heat and mass transfer in both reacting and nonreacting systems. Hayday (*7*) cites most of the existing literature dealing with such nonreacting systems, and describes a general class of similar flows engendered by spinning bodies of revolution. Litt and Serad (*9*) treat the rotating disk system with chemical reactions and discuss pertinent literature.

ANALYSIS

In the face of extensive existing literature treating rotating disk systems, it is unnecessary to discuss the defining equations in detail here. Rather, the equations developed elsewhere (*17*) will be used as a starting point, with slight changes. The coordinate system is sketched in Figure 1. By using the independent variable

$$\eta = Z\left(\frac{\omega}{\nu}\right)^{1/2}, \quad \omega = \text{angular velocity}, \quad \nu = \text{kinematic viscosity} \tag{1}$$

and the following dependent variables

$$F(\eta) = \frac{V_r}{r\omega}, \quad G(\eta) = \frac{V_\theta}{r\omega}, \quad H(\eta) = \frac{V_z}{(\omega\nu)^{1/2}} \tag{2}$$

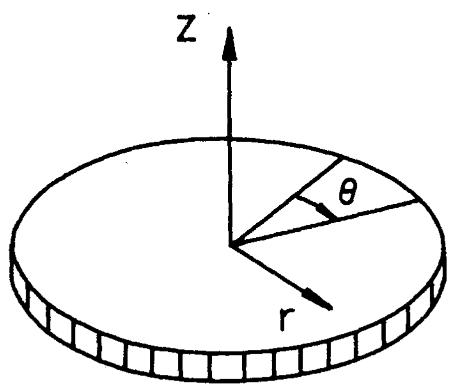

Fig. 1. Coordinate system for rotating disk.

$$P(\eta) = \frac{p}{\mu\omega}, \quad \phi(\eta) = \frac{w_s}{w_{sw}}$$

the conservation equations for constant property flow are

momentum: $$F'' = HF' + F^2 - G^2 \tag{3a}$$

$$G'' = HG' + 2FG \tag{3b}$$

$$P' = H'' - HH' \tag{3c}$$

continuity: $$H' = -2F \tag{4}$$

diffusion: $$\phi'' = N_{Sc} H \phi' \tag{5}$$

Clearly, Equations (3a), (3b), (4) and (5) are independent of Equation (3c), so the latter may be ignored. Further, Equation (4) may be integrated and substituted into Equations (3a) and (3b) to yield

$$F'' = F^2 + F'\left(H_w - 2\int_0^\eta F d\eta \right) - G^2 \tag{6}$$

$$G'' = 2FG + G'\left(H_w - 2\int_0^\eta F d\eta \right) \tag{7}$$

The appropriate boundary conditions are

$$\text{at } Z = 0: \quad \left. \begin{array}{l} V_r = 0 \\ V_\theta = r\omega \end{array} \right\} \text{no-slip conditions}$$

$$V_z = V_{z_w}, \quad w_s = w_{sw} \tag{8}$$

$$\text{at } Z \to \infty: \quad V_r \to 0, \quad V_\theta \to 0, \quad w_s \to w_{s\infty}$$

or, in terms of the transformed variables,

$$\text{at } \eta = 0: \quad F = 0, \quad G = 1, \quad H = H_w, \quad \phi = 1$$

$$\text{at } \eta \to \infty: \quad F \to 0, \quad G \to 0, \quad \phi \to \phi_\infty \tag{9}$$

The method used in the present study to solve Equations (6) and (7) is believed to be different from any described in the literature, and has application in other problems also, so it will be described in some detail here. Basically, the problem is to find $F'(0)$ and $G'(0)$ such that upon integrating Equations (6) and (7), and by using $F(0) = 0$, $G(0) = 1$ and $H(0) = H_w$, the conditions $F(\infty) = G(\infty) = 0$ are satisfied. To do this, let

$$F(\eta) = F[\eta, F'(0), G'(0)]$$

$$G(\eta) = G[\eta, F'(0), G'(0)]$$

Denote guesses of $F'(0)$ by A_i, guesses of $G'(0)$ by B_i. Then, approximating F and G by the first three terms of Taylor series expansions

$$0 = F(\infty) \cong F(\infty, A_1, B_1) + \left.\frac{\partial F}{\partial F'(0)}\right|_\infty (F'(0) - A_1) + \left.\frac{\partial F}{\partial G'(0)}\right|_\infty (G'(0) - B_1) \tag{10}$$

$$0 = G(\infty) \cong G(\infty, A_1, B_1) + \left.\frac{\partial G}{\partial F'(0)}\right|_\infty (F'(0) - A_1) + \left.\frac{\partial G}{\partial G'(0)}\right|_\infty (G'(0) - B_1) \tag{11}$$

If $\left.\frac{\partial F}{\partial F'(0)}\right|_\infty, \left.\frac{\partial F}{\partial G'(0)}\right|_\infty, \left.\frac{\partial G}{\partial F'(0)}\right|_\infty, \left.\frac{\partial G}{\partial G'(0)}\right|_\infty$ are known, Equations (10) and (11) can be solved for new guesses of $F'(0)$ and $G'(0)$. The more or less standard method is to solve Equations (6) and (7) for three sets of (A_i, B_i), and to use the resulting values of $F(\infty)$ and $G(\infty)$ to obtain approximate values of $\left.\frac{\partial F}{\partial F'(0)}\right|_\infty$ etc. by finite difference. However, exact values of $\left.\frac{\partial F}{\partial F'(0)}\right|_\infty$, etc., for a given set (A_i, B_i) may be obtained as follows. Let

$$s = \frac{\partial G}{\partial F'(0)} \qquad u = \frac{\partial F}{\partial F'(0)}$$

$$t = \frac{\partial G}{\partial G'(0)} \qquad v = \frac{\partial F}{\partial G'(0)}$$

Equations (6) and (7) are then differentiated by $F'(0)$ to yield

$$u'' = 2Fu - 2F'\int_0^\eta u d\eta + u'\left(H_w - 2\int_0^\eta F d\eta \right) - 2Gs \tag{12}$$

$$s'' = 2Fs + 2uG - 2G'\int_0^\eta u d\eta + s'\left(H_w - 2\int_0^\eta F d\eta \right) \tag{13}$$

and then differentiated by $G'(0)$ to yield

$$v'' = 2Fv - 2F'\int_0^\eta v d\eta + v'\left(H_w - 2\int_0^\eta F d\eta \right) - 2Gt \tag{14}$$

$$t'' = 2Ft + 2vG - 2G'\int_0^\eta v d\eta + t'\left(H_w - 2\int_0^\eta F d\eta \right) \tag{15}$$

The boundary conditions for Equations (12) to (15) are

$$s(0) = t(0) = u(0) = v(0) = s'(0) = v'(0) = 0 \tag{16}$$

$$t'(0) = u'(0) = 1$$

One then assumes values of A_1 and B_1 and solves Equations (6) and (7). These results are then used to solve Equations (12) to (15), thereby obtaining values of $s(\infty)$, $t(\infty)$, $u(\infty)$ and $v(\infty)$ which may be used in Equations (10) and (11) to obtain new guesses of $F'(0)$ and $G'(0)$. The process is repeated until convergence is obtained.

Litt and Serad (9) have indicated that they obtained solutions to the rotating disk equations with relative ease on an analog computer. A major advantage of the present method is that it can be applied to variable property problems, which would be enormously difficult on an analog computer.

Solution of Equation (5) is straightforward; two integrations yields

$$\phi = 1 + \phi'(0)\int_0^\eta e^{N_{Sc}\int_0^\eta H\, d\eta} d\eta \tag{17}$$

where $H = H_w - 2\int_0^\eta F d\eta$ is known from the solution to Equations (6) and (7). There are two additional boundary conditions to be considered in the present reverse osmosis problem. The first arises from a simple mass balance at the disk surface; the mass flux of salt in the Z-direction is given by

$$n_s = w_s \rho V_z - \rho D \frac{\partial w_s}{\partial z}$$

so that at the wall we have

$$\frac{D}{w_{sw}} \left.\frac{\partial w_s}{\partial z}\right|_w = V_{zw} - \frac{n_{sw}}{\rho\, w_{sw}}$$

$$= \left[1 - \frac{n_{sw}}{w_{sw}\rho V_{zw}}\right] V_{zw} = R\ V_{zw} \quad (18)$$

The parameter R is a measure of the salt rejection, and is independent of r and θ. For an ideal membrane, $n_{sw} = 0$, $R = 1$. In terms of the transformed variables, Equation (18) becomes

$$\phi'(0) = RN_{Sc}H_w \quad (19)$$

A further condition arises from the common assumption that the flow of water through the membrane may be described by

$$-(1 - w_{sp})\, V_{zw} = A[\Delta P - \Delta\pi]$$

$$= A\Delta P \left[1 - B\left(\frac{w_{sw} - w_{sp}}{w_{s\infty}}\right)\right]$$

where A is a constant characteristic of a particular membrane, π_0 is the osmotic pressure of the solution at its original salinity, ΔP is the pressure difference across the membrane, and $B = \pi_0/\Delta P$.* This equation transforms to

$$H_w = \frac{V_{zw}}{(\omega\nu)^{1/2}} = -\frac{A\Delta P}{(\omega\nu)^{1/2}(1 - w_{sp})}\left[1 - \frac{BR}{\phi_\infty}\right] \quad (20)$$

By combining Equations (17) and (19), one obtains

$$\phi_\infty = 1 + RN_{Sc}H_w \int_0^\infty e^{N_{Sc}\int_0^\eta H\, d\eta}\, d\eta \quad (21)$$

As pointed out (*18*), one need only obtain solutions to Equation (21) for $R = 1$, since this equation may be rewritten as

$$\phi_\infty = 1 + R(\phi_\infty|_{R=1} - 1) \quad (22)$$

This equation can also be used to develop a useful relation for R:

$$R = 1 - \frac{n_{sw}}{w_{sw}\rho V_{zw}} = 1 - \frac{n_{sw}}{n_{tw}w_{sw}} = 1 - \frac{w_{sp}}{w_{s\infty}}\phi_\infty$$

$$= 1 - \frac{w_{sp}}{w_{s\infty}}[1 - R(1 - \phi_\infty|_{R=1})]$$

where w_{sp} is the mass fraction of salt in the product water. The solution for R is

$$R = \frac{1 - \dfrac{w_{sp}}{w_{s\infty}}}{1 - \dfrac{w_{sp}}{w_{s\infty}}(1 - \phi_\infty|_{R=1})} \quad (23)$$

Hence this system, like the stagnation flow case discussed elsewhere (*18*), should be quite useful in determining the functional dependence of R on water flux and ΔP through Equations (22) and (23).

Equations (6), (7) and (21) have been solved exactly for $R = 1$ and $N_{Sc} = 560$ for a wide range of values of H_w; the results are shown in Table 1 along with some approximate solutions which will be described next.

TABLE 1. COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS TO THE DIFFUSION EQUATION FOR THE ROTATING DISK SYSTEM WITH $N_{Sc} = 560$.

$-H_w$	$-H''(0)$	ϕ_∞ Exact	ϕ_∞ Equation (31) $K, K^2 \equiv 0$ $H''(0) \equiv -1.02$	n*	ϕ_∞ Equation (31) $K^2 \equiv 0$ $H''(0) \equiv -1.02$	n*
0	1.0204	1.0000				
0.0005			0.947	2	0.945	2
0.001			0.897	2	0.894	2
0.005			0.590	4	0.580	4
0.010	1.0194	.3500	0.361	6	0.350	6
0.0125			0.287	8	0.277	8
0.015	1.0189	.2201	0.229	9	0.220	9
0.0175	1.0187	.1771	0.185	10	0.177	11
0.02	1.0184	.1437	0.151	12	0.144	13
0.025	1.0180	.0971	0.102	15	0.097	16
0.03	1.0175	.0676	0.0714	18	0.0679	19
0.035	1.0169	.0484	0.0515	22	0.0490	23
0.04			0.0379	25	0.0355(+)	25

* Number of terms required to converge to accuracy given.

APPROXIMATE SOLUTION

Approximate solutions to equations similar to Equation (21) have been obtained by many workers (*1, 8, 10, 15, 16, 18*). Significant improvements can be made on these solutions however, which make them applicable for a wider class of geometries, for lower Prandtl and Schmidt numbers, and for a wide range of interfacial mass transfer rates. The analysis is straightforward and will be only briefly sketched before applying the results to the rotating disk systems.

Consider the equation

$$\Lambda'' + \sigma E\, \Lambda' = 0; \quad \Lambda(0) = 1; \quad \Lambda'(0) = \Lambda'(0)$$

$$\text{or} \quad \Lambda(\infty) = \Lambda_\infty \quad (24)$$

which may be integrated to give

$$\Lambda_\infty = 1 + \Lambda'(0)\int_0^\infty e^{-\sigma\int_0^\eta E\, d\eta}\, d\eta \quad (25)$$

This equation is representative of the energy

$$\left(\Lambda = \frac{T}{T_w} \text{ or } \frac{T - T_\infty}{T_w - T_\infty}\right)$$

* Note that the induced pressure p is small everywhere and is in fact zero at the disk surface (*12*) so ΔP is exactly the pressure difference between the ambient stream and the water behind the membrane.

or diffusion

$$\left(\Lambda = \frac{w}{w_w} \text{ or } \frac{w - w_\infty}{w_w - w_\infty} \right)$$

equation for $\sigma = N_{Pr}$ or N_{Sc} respectively, for wedge-type flows with E as the dimensionless stream function or for the rotating disk with $E = -H$. This equation also applies for the momentum equation in the case of a flat plate, again with E as the dimensionless stream function,† and

$$\Lambda = 1 - u/u_\infty = 1 - E'$$

Assume now that E may be approximated by

$$E = E_w + \frac{E''(0)}{2}\eta^2 + \frac{E'''(0)}{6}\eta^3 \qquad (26)$$

The most accurate analysis (*15*) of the above cited references presented results in tabular form for a wide range of values of $E'''(0)$ and E_w, including suction and blowing. A closed form approximate solution that retains most of the accuracy attained by Spalding and Evans may be obtained as follows.

Substitute Equation (26) into Equation (25), expand the resulting exponential factors involving E_w and $E'''(0)$ in power series, retaining only the first three terms of the latter. The result is

$$\Lambda_\infty = 1 + \Lambda'(0) \left[\frac{6}{E''(0)\sigma} \right]^{1/3} \sum_{j=0}^{\infty} J^j\, a_j\, (1 + b_j K + c_j K^2)$$

$$a_j = \Gamma\left(\frac{j+1}{3}\right) / 3\Gamma(j+1),$$

$$b_j = \Gamma\left(\frac{j+5}{3}\right) / \Gamma\left(\frac{j+1}{3}\right)$$

$$c_j = \Gamma\left(\frac{j+9}{3}\right) / 2\Gamma\left(\frac{j+1}{3}\right),$$

$$J = -\sigma^{2/3} E_w [6/E''(0)]^{1/3}$$

$$K = -\frac{E'''(0)}{24\sigma^{1/3}} [6/E''(0)]^{4/3} \qquad (27)$$

Since Equation (27) is significantly more accurate than closed form solutions previously obtained, it is of value to indicate its application to a few boundary layer problems. Consider first the momentum equation for flow over a flat plate, wherein $\sigma = 1$ and $\Lambda = 1 - E'$. In this case $E'''(0) \sim E^{IV}(0) \sim 0$. By ignoring these terms then, but extending the approximation in Equation (26) to include the $E^{V}(0)$ term, following the same procedure outlined above for the $E'''(0)$ term, and retaining only the first two terms in the expansion in J, there results

$$E''(0) = \{0.474\, E_w + \sqrt{0.225\, E_w^2 + 0.604}\}^3 \quad (28)$$

For $E_w = 0$, this gives $E''(0) = 0.469$ which is very near the exact result of 0.4696. Comparison with exact solutions in Table 2 show Equation (28) to be valid for a wide range of suction and blowing. Also presented in this table are values of $\Lambda'(0)$ calculated from Equation (27) for $\sigma = 1$, using exact values of $E''(0)$ shown in the same table. Clearly the results are seen to be quite good, indicating that Equation (27) will be accurate for obtaining solutions to the energy and diffusion equations for σ ($= N_{Pr}$ or N_{Sc}) ≥ 1. The accuracy of this equation can be further improved by retaining the $E^{V}(0)$ term as above, which leads to

$$a_j = \left[\Gamma\left(\frac{j+1}{3}\right) + \frac{1}{20}\Gamma\left(\frac{j+7}{3}\right) \right] / 3\Gamma(j+1)$$

in place of the a_j defined previously. The K and K^2 terms will be comparatively small.

Table 2. Comparison of Exact and Approximate Solutions to the Momentum Equations for the Flat Plate System

$F(0)$	$E''(0)$ Exact (*16*)	$E''(0)$ Eq. (28)	$-\Lambda'(0)$ Eq. (27) $\sigma = 1$	$-\Lambda'(0)$ Eq. (27) $\sigma = 1$; $K, K^2 = 0$
0.4950	0.8538		0.853	0.899
0.4243	0.7962		0.797	0.837
0.3536	0.7394		0.742	0.776
0.2828	0.6834		0.688	0.715
0.2121	0.6284	0.691	0.635	0.655
0.1414	0.5743	0.608	0.582	0.596
0.0707	0.5214	0.534	0.530	0.537
0	0.4696	0.469	0.479	0.479
−0.0707	0.4191	0.412		
−0.1414	0.3700	0.363	0.379	0.366
−0.2121	0.3225	0.319	0.330	0.312
−0.2828	0.2766	0.281	0.281	0.259
−0.3536	0.2326	0.247	0.232	0.209
−0.4243	0.1907		0.181	0.161
−0.4950	0.1512	0.192	0.129	0.116

As another example‡, wherein the K and K^2 terms in Equation (27) are more important, consider the energy and diffusion equations for the case of plane stagnation flow. For $E_w = 0$, $E''(0) = 1.2326$, $E'''(0) = -1$, so Equation (27) gives

$$\Lambda'(0) = -0.661\,\sigma^{1/3}\,[1 + 0.116\,\sigma^{-1/3} + 0.044\,\sigma^{-2/3}]^{-1} \qquad (29)$$

and for $\sigma = 1$, $\Lambda'(0) = -0.570$ which is very close to the exact value, −0.5704. It should be noted that the result obtained if $K = 0$, that is, $\Lambda'(0) = -0.661\,\sigma^{1/3}$, is equivalent to the result obtained by Lighthill (*8*) which is in error by ~ 16% for $\sigma = 1$. Again, comparison with exact solutions shows Equation (27) to be valid for a wide range of suction and blowing.

As a final example, consider the rotating disk system, for $E_w = 0$, $E''(0) = 1.0204$, $E'''(0) = 2$. Equation (27) gives

$$\Lambda'(0) = -0.620\,\sigma^{1/3}\,[1 + 0.298\,\sigma^{-1/3} + 0.292\,\sigma^{-2/3}]^{-1} \qquad (30)$$

For $\sigma = 1$, this equation yields $\Lambda'(0) = -0.390$ which is very close to the exact result, −0.3963. Equation (30) is a vast improvement over the result obtained by Litt and Serad (*9*) that is, $\Lambda'(0) = -0.620\,\sigma^{1/3}$, which is in error by 57% for $\sigma = 1$.

The approximate solution developed herein and represented by Equation (27) is therefore a significant improvement over existing closed form approximate solutions. It is accurate for a very broad range of conditions, for wedge flows, from the flat plate to stagnation flow, for σ from 1 (or possibly less) to ∞, and for a wide range of suction and blowing.

We now turn attention to the application of the approximate solution to the case of reverse osmosis in rotating disk systems. The preceding discussion was based on use of the boundary condition $\Lambda_\infty = 0$. For the reverse osmosis case, however, we will take $\Lambda = \phi$ as defined in Equation (2). Hence $\Lambda_\infty \neq 0$, and the prescribed boundary condition is that given by Equation (19), that is

$$\Lambda'(0) = \phi'(0) = R N_{Sc} H_w$$

† F is usually used to denote the dimensionless stream function. E is used here to avoid confusion with F defined in Equation (2).

‡ In these examples we assumed Λ equals $T - T_\infty/T_w - T_\infty$ or $w_A - w_{A\infty}/w_{A_w} - w_{A\infty}$ so $\Lambda_\infty = 0$.

Inserting this into Equation (27) there results

$$\phi_\infty = 1 + \sum_{j=0}^{\infty} J^{j+1}\, a_j(1 + K\, b_j + K^2\, c_j) \qquad (31)$$

where

$$J = N_{Sc}^{2/3}\, H_w \left[\frac{6}{-H''(0)}\right]^{1/3}$$

$$K = \frac{H'''(0)}{24} N_{Sc}^{-1/3} \left[\frac{6}{-H''(0)}\right]^{4/3}$$

$$= \frac{2 + H_w H''(0)}{24} N_{Sc}^{-1/3} \left[\frac{6}{-H''(0)}\right]^{4/3}$$

As pointed out earlier, solutions for arbitrary R may be obtained from those for $R = 1$, so R has been set equal to 1 in Equation (31). As may be seen in Table 1, $H''(0)$ does not vary significantly for conditions of interest. Hence it is reasonable to use a constant value of $H''(0)$ equal to -1.02 in Equation (31). In this case

$$J = 1.803\ N_{Sc}^{2/3}\ H_w$$

$$K = 0.883\ N_{Sc}^{-1/3}\,[1 - 0.5103\ H_w]$$

Approximate solutions obtained using these relations in Equation (31) are presented in Table 1, for $Sc = 560$ which is appropriate for dilute saline systems, as calculated by using only the first, and the first and second terms in parentheses. Both forms are seen to be very accurate, indicating that for all practical purposes the K^2 and K terms can be ignored in this high Schmidt number range.

APPLICATIONS OF SOLUTIONS TO REVERSE OSMOSIS SYSTEMS

As in the case of stagnation flow treated in an earlier paper (*18*), the solutions obtained for reverse osmosis in rotating disk systems provide a means of experimentally determining the parameters A and R for a membrane, or, if known, of specifying a system.

For example, suppose it is desired to determine experimentally the values of A and R for a membrane. For a given run one would measure ΔP, V_{z_w}, w_{s_p} and ω. The value of H_w follows from Equation (2), and $\phi_\infty|_{R=1}$ can be determined form Equation (31) or, for $N_{Sc} = 560$, from Table 1. R is then evaluated from Equation (23), and ϕ_∞ obtained from Equation (22). Finally, A is then obtained from Equation (20). Note that the rather difficult task of measuring w_{s_w} is avoided, and that this value arises as a result of the analysis.

Suppose, on the other hand, A and R are known, and we wish to specify the pure water production $(1 - w_{s_p})\,V_{z_w}$. Equation (20) then provides a direct relation between B and ϕ_∞. Selecting a feasible operating pressure, that is, selecting B, one obtains a corresponding value of ϕ_∞. Since R is known, H_w is then specified, and consequently so is ω. Clearly, $\phi_\infty|_{R=1}$ is also known, and Equation (23) can be used to estimate the mass fraction of salt that will appear in the product water. There are, of course, alternate ways to specify the system.

CONCLUSION

The reverse osmosis problem in rotating disk systems has been solved both exactly and with a very accurate approximate solution, and the analysis should be especially useful in the experimental determination of membrane properties. The approximate solutions have been shown to be applicable to a broad variety of systems and conditions.

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NOTATION

A = membrane constant defined by Equation (20)
B = $\pi_0/\Delta P$
D = diffusion coefficient
E = dummy variable
F, G, H = dimensionless velocity variables defined by Equation (2)
n_s = mass flux of salt, defined by Equation (18)
n_t = total mass flux through membrane
p = pressure
ΔP = pressure drop across membrane
r = radial coordinate
R = measure of salt rejection by membrane, defined by Equation (18)
N_{Sc} = Schmidt number, ν/D
V = velocity
w_s = mass fraction of salt in saline solution
Z = coordinate perpendicular to disk

Greek Symbols

ρ = density
π_0 = osmotic pressure of original saline solution
ϕ = w_s/w_{sw}
ν = kinematic viscosity
η = dimensionless coordinate, $Z(\omega/\nu)^{1/2}$
ω = angular velocity
θ = angular coordinate
Γ = gamma function
μ = viscosity
Λ = dummy variable

Subscripts

∞ = value in ambient fluid
p = value in product water
w = value at membrane surface
r = radial component
Z = axial component
θ = angular component

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